

Modules over the Heisenberg-Virasoro and $W(2, 2)$ algebras

Hongjia Chen and Xiangqian Guo

Abstract

In this paper, we consider the modules for the Heisenberg-Virasoro algebra and the W algebra $W(2, 2)$. We determine the modules whose restriction to the Cartan subalgebra (modulo center) are free of rank 1 for these two algebras. We also determine the simplicity of these modules. These modules provide new simple modules for the W algebra $W(2, 2)$.

Keywords: Heisenberg-Virasoro algebra, W algebra $W(2, 2)$, simple module.

2000 Math. Subj. Class.: 17B10, 17B20, 17B65, 17B66, 17B68

1 Introduction

Let $\mathbb{C}, \mathbb{Z}, \mathbb{Z}_+, \mathbb{N}$ be the sets of all complexes, all integers, all non-negative integers and all positive integers, respectively. The **Virasoro algebra** Vir is an infinite dimensional Lie algebra over the complex numbers \mathbb{C} , with basis $\{L_n, C \mid n \in \mathbb{Z}\}$ and defining relations

$$[L_m, L_n] = (n - m)L_{n+m} + \delta_{n, -m} \frac{m^3 - m}{12} C, \quad m, n \in \mathbb{Z},$$

$$[C, L_m] = 0, \quad m \in \mathbb{Z},$$

which is the universal central extension of the so-called infinite dimensional **Witt algebra** of rank 1.

The theory of weight Virasoro modules with finite-dimensional weight spaces is fairly well developed (see [KR] and references therein). In 1992, O. Mathieu [M] classified all simple Harish-Chandra modules, that is, simple modules with finite-dimensional weight spaces, over the Virasoro algebra, which is conjectured by Kac [K]. Recently, many authors constructed many classes of simple non-Harish-Chandra modules, including simple weight modules with infinite-dimensional weight spaces (see [CGZ, CM, LLZ, LZ2, Z]) and simple non-weight modules (see [BM, LGZ, LLZ, LZ1, MW, MZ, OW, TZ1, TZ2, TZ3, TZ4]).

For $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $\alpha \in \mathbb{C}$, denote by $\Omega(\lambda, \alpha) = \mathbb{C}[t]$ the polynomial algebra over \mathbb{C} . In [LZ1] the Vir-module structure on $\Omega(\lambda, \alpha)$ is given by

$$Ct^i = 0, L_mt^i = \lambda^m(t - ma)(t - m)^i, \quad m \in \mathbb{Z}, i \in \mathbb{Z}_+.$$

From [LZ1] we know that $\Omega(\lambda, \alpha)$ is simple if and only if $\lambda, \alpha \in \mathbb{C}^*$. If $\alpha = 0$, then $\Omega(\lambda, 0)$ has an simple submodule $t\Omega(\lambda, 0)$ with codimension 1.

Recently Tan and Zhao [TZ4] showed that the above defined modules $\Omega(\lambda, \alpha)$ are just those Vir-modules that are free of rank 1 when restricted to the subalgebra $\mathbb{C}L_0$. In fact

they did similar work for the Witt algebras of all ranks: classifying all Witt algebra modules that are free of rank 1 when restricted to the Cartan subalgebra of the Witt algebra. This kind of problems originated from an earlier work [N] of J. Nilson who determined the \mathfrak{sl}_n -modules that are free of rank 1 when restricted to the Cartan subalgebra of \mathfrak{sl}_n , where $n \geq 2$ is a positive integer.

In the present paper, we will follow their ideas to consider similar questions for the Heisenberg-Virasoro algebra and the W algebra $W(2, 2)$. The Heisenberg-Virasoro algebra was introduced and studied in [ACKP] and for more results for this algebra please refer to [B, LZ3, CG] and references therein. The W algebra $W(2, 2)$ was first introduced and studied in [ZD] and for more information about this algebras please refer to [GLZ, LiZ].

2 Representations over the Heisenberg-Virasoro algebra

The **Heisenberg-Virasoro algebra** \mathcal{L} is the universal central extension of the Lie algebra $\{f(t)\frac{d}{dt} + g(t) \mid f, g \in \mathbb{C}[t^{\pm 1}]\}$ of differential operators of order at most one. More precisely, it is the complex Lie algebra that has a basis $\{L_n, I_n, C_1, C_2, C_3 \mid n \in \mathbb{Z}\}$ subject to the following Lie brackets:

$$\begin{aligned} [L_m, L_n] &= (n - m)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C_1 \\ [L_m, I_n] &= nI_{m+n} + \delta_{m+n,0}(m^2 + m)C_2 \\ [I_m, I_n] &= m\delta_{m+n,0}C_3 \end{aligned} \quad \forall m, n \in \mathbb{Z}$$

where C_1, C_2, C_3 and I_0 span the center of \mathcal{L} . The Lie subalgebra spanned by $\{L_n, C_1 \mid n \in \mathbb{Z}\}$ is just the Virasoro algebra.

In this section we will determine the Heisenberg-Virasoro algebra modules which are free of rank 1 when regarded as a $\mathbb{C}L_0$ -module. For convenience, we cite the following result in [TZ4].

Theorem 1. Let M be a $U(\text{Vir})$ -module such that the restriction of $U(\text{Vir})$ to $U(\mathbb{C}L_0)$ is free of rank 1, that is, $M = U(\mathbb{C}L_0)v$ for some torsion-free $v \in M$. Then $M \cong \Omega(\lambda, \alpha)$ for some $\alpha \in \mathbb{C}, \lambda \in \mathbb{C}^*$. Moreover, M is simple if and only if $M \cong \Omega(\lambda, \alpha), \lambda, \alpha \in \mathbb{C}^*$.

The Vir-module $\Omega(\lambda, \alpha)$ can be naturally made into an \mathcal{L} -module.

Example 1. For $\lambda \in \mathbb{C}^*, \alpha, \beta \in \mathbb{C}$, denote by $\Omega(\lambda, \alpha, \beta) = \mathbb{C}[t]$ the polynomial algebra over \mathbb{C} . We can define the \mathcal{L} -module structure on $\Omega(\lambda, \alpha, \beta)$ as follows

$$C_j f(t) = 0, \quad L_m f(t) = \lambda^m(t - m\alpha)f(t - m), \quad I_m f(t) = \beta\lambda^m f(t - m)$$

where $f(t) \in \mathbb{C}[t], m \in \mathbb{Z}$ and $j = 1, 2, 3$. From [LZ1] it is easy to know that $\Omega(\lambda, \alpha, \beta)$ is simple if and only if $\alpha \neq 0$ or $\beta \neq 0$. If $\alpha = \beta = 0$, then $\Omega(\lambda, 0, 0)$ has a simple submodule $t\Omega(\lambda, 0, 0)$ with codimension 1.

Now we have the main result of this section:

Theorem 2. Let M be a $U(\mathcal{L})$ -module such that the restriction of $U(\mathcal{L})$ to $U(\mathbb{C}L_0)$ is free of rank 1. Then $M \cong \Omega(\lambda, \alpha, \beta)$ for some $\alpha, \beta \in \mathbb{C}, \lambda \in \mathbb{C}^*$. Moreover M is simple if and only if $M \cong \Omega(\lambda, \alpha, \beta)$ for some $\lambda \in \mathbb{C}^*, \alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$ or $\beta \neq 0$.

Proof. The second result follows from the first result and Example 1. We now prove the first result.

Denote the action of I_0, C_1, C_2, C_3 by c_0, c_1, c_2, c_3 , respectively. By Theorem 1, we have $M \cong \Omega(\lambda, \alpha) = \mathbb{C}[t]$ as Vir-modules, where $\lambda, \alpha \in \mathbb{C}, \lambda \neq 0$ and t is an indeterminant. Hence we can assume that $c_1 = 0$ and

$$L_m f(t) = \lambda^m (t - m\alpha) f(t - m), \quad m \in \mathbb{Z}, f(t) \in \mathbb{C}[t].$$

Now we consider the action of I_m 's. First we can easily get that

$$I_m(f(t)) = I_m(f(L_0)(1)) = f(t - m)I_m(1), \quad \forall m \in \mathbb{Z}. \quad (2.1)$$

Case 1. $I_m(1) = 0$ for some $m \in \mathbb{Z}^*$.

By (2.1) we know that $I_m(M) = 0$. Now following from the defining relations of \mathcal{L} , we have $I_n(M) = 0$ for all $n \in \mathbb{Z}^*$ and $c_0 = c_2 = c_3 = 0$. In this case $M \cong \Omega(\lambda, \alpha, 0)$.

Case 2. $I_m(1) \neq 0$ for all $m \in \mathbb{Z}^*$.

Assume that

$$I_m(1) = f_m(t) = \sum_{i=0}^{k_m} b_{m,i} t^i, \quad \forall m \in \mathbb{Z},$$

where $b_{m,i} \in \mathbb{C}$ and $b_{m,k_m} \neq 0$. Now we calculate $[L_{-m}, I_m](1)$ as follows

$$\begin{aligned} [L_{-m}, I_m](1) &= L_{-m}I_m(1) - I_mL_{-m}(1) \\ &= L_{-m}f_m(t) - I_m\lambda^{-m}(t + m\alpha) \\ &= \lambda^{-m}(t + m\alpha)f_m(t + m) - \lambda^{-m}(t - m + m\alpha)f_m(t) \\ &= \lambda^{-m}m(k_m + 1)b_{m,k_m}t^{k_m} + \text{lower-degree terms w.r.t. } t, \end{aligned}$$

which implies that $k_m = 0$ for all $m \in \mathbb{Z}$. Hence we have $I_m(1) = a_m \in \mathbb{C}$ for all $m \in \mathbb{Z}$ with $a_0 = c_0$. Now $[I_m, I_n](1) = m\delta_{m+n,0}c_3$ implies that

$$c_3 = [I_1, I_{-1}](1) = a_1a_{-1} - a_{-1}a_1 = 0 \quad (2.2)$$

and $[L_m, I_n](1) = nI_{m+n}(1) + \delta_{m+n,0}(m^2 + m)c_2$ implies

$$na_{m+n} + \delta_{m+n,0}(m^2 + m)c_2 = a_n\lambda^m(t - m\alpha) - a_n\lambda^m(t - n - m\alpha) = na_n\lambda^m. \quad (2.3)$$

Taking $m = -1$ in (2.3), we have $a_n = \lambda a_{n-1}$ for all $n \neq 0$. Hence $c_2 = 0$. Then checking the recurrence relations $na_{m+n} = na_n\lambda^m$, we have that $a_m = c_0\lambda^m$.

The above discussion allows us to establish an \mathcal{L} -module homomorphism

$$M \longrightarrow \Omega(\lambda, \alpha, c_0), \quad t^i \mapsto t^i.$$

Clearly, this is an \mathcal{L} -module isomorphism. This completes the proof. \square

3 Representations over the W -algebra $W(2, 2)$

Let \mathcal{W} denote the complex **Lie algebra** $W(2, 2)$, that has a basis $\{L_n, W_n, C_1, C_2 \mid n \in \mathbb{Z}\}$ and the Lie brackets defined by:

$$\begin{aligned} [L_m, L_n] &= (n - m)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C_1 \\ [L_m, W_n] &= (n - m)W_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C_2 \\ [W_m, W_n] &= 0 \end{aligned}$$

where C_1 and C_2 are central in \mathcal{W} . The Lie subalgebra spanned by $\{L_n, C_1 \mid n \in \mathbb{Z}\}$ is the Virasoro algebra.

3.1 Rank 1 free $U(\mathbb{C}L_0)$ -modules. The Vir-module $\Omega(\lambda, \alpha)$ can be made into a \mathcal{W} -module:

Example 2. For $\lambda \in \mathbb{C}^*, \alpha \in \mathbb{C}$, denote by $\Omega_{\mathcal{W}}(\lambda, \alpha) = \mathbb{C}[t]$ the polynomial algebra over \mathbb{C} . We can define the \mathcal{W} -module structure on $\Omega_{\mathcal{W}}(\lambda, \alpha)$ as follows

$$C_j f(t) = W_m f(t) = 0, \quad L_m f(t) = \lambda^m (t - m\alpha) f(t - m)$$

where $f(t) \in \mathbb{C}[t]$, $m \in \mathbb{Z}$ and $j = 1, 2$. From [LZ1] it is easy to know that $\Omega_{\mathcal{W}}(\lambda, \alpha)$ is simple if and only if $\alpha \neq 0$. If $\alpha = 0$, then $\Omega_{\mathcal{W}}(\lambda, 0)$ has an simple submodule $t\Omega_{\mathcal{W}}(\lambda, 0)$ with codimension 1.

In this subsection, we will classify the modules over the Lie algebra \mathcal{W} such that the modules are free $U(\mathbb{C}L_0)$ -modules of rank 1. We have

Theorem 3. Let M be a $U(\mathcal{W})$ -module such that the restriction of $U(\mathcal{W})$ to $U(\mathbb{C}L_0)$ is free of rank 1. Then $M \cong \Omega_{\mathcal{W}}(\lambda, \alpha)$ for some $\alpha \in \mathbb{C}, \lambda \in \mathbb{C}^*$. Moreover, M is simple if and only if $M \cong \Omega_{\mathcal{W}}(\lambda, \alpha), \lambda, \alpha \in \mathbb{C}^*$.

Proof. Denote the action of C_1, C_2 by c_1, c_2 , respectively. By Theorem 1, $M \cong \Omega(\lambda, \alpha) = \mathbb{C}[t]$ as Vir-modules for some $\lambda \in \mathbb{C}^*$ and $\alpha \in \mathbb{C}$. Hence we can assume that $c_1 = 0$ and

$$L_m f(t) = \lambda^m (t - m\alpha) f(t - m), m \in \mathbb{Z}, f(t) \in \mathbb{C}[t].$$

Now we consider the action of W_m 's. First we can easily get that

$$W_m(f(t)) = W_m(f(L_0)(1)) = f(t - m)W_m(1), \quad \forall m \in \mathbb{Z}. \quad (3.1)$$

Case 1. $W_m(1) = 0$ for some $m \in \mathbb{Z}$.

By (3.1) we know that $W_m(M) = 0$. Now following from the definition relations of \mathcal{W} , we have $W_n(M) = 0$ for all $n \in \mathbb{Z}$ and $c_2 = 0$. We see that M is actually a Vir-module and hence $M \cong \Omega_{\mathcal{W}}(\lambda, \alpha)$ in this case.

Case 2. $W_m(1) \neq 0$ for all $m \in \mathbb{Z}$.

Assume that

$$W_m(1) = f_m(t) = \sum_{i=0}^{k_m} b_{m,i} t^i$$

where $b_{m,i} \in \mathbb{C}$ and $b_{m,k_m} \neq 0$. For all $mn \leq 0$, we have

$$\begin{aligned} 0 &= [W_m, W_n](1) \\ &= W_m W_n(1) - W_n W_m(1) \\ &= W_m f_n(t) - W_n f_m(t) \\ &= f_n(t-m) f_m(t) - f_n(t) f_m(t-n) \\ &= b_{m,k_m} b_{n,k_n} (n k_m - m k_n) t^{k_m+k_n-1} + \text{lower-degree terms w.r.t. } t. \end{aligned}$$

Hence $k_m = 0$ for all $m \in \mathbb{Z}$, i.e., $W_m(1) = a_m$ for some $a_m \in \mathbb{C}^*$. Now

$$[L_m, W_n](1) = (n-m)W_{m+n}(1) + \delta_{m+n,0} \frac{m^3-m}{12} c_2$$

implies

$$(n-m)a_{m+n} + \delta_{m+n,0} \frac{m^3-m}{12} c_2 = a_n \lambda^m (t-m\alpha) - a_n \lambda^m (t-n-m\alpha) = n a_n \lambda^m. \quad (3.2)$$

Taking $m = -1$ and $m = 1$ in (2.3) respectively, we have

$$(n+1)a_{n-1} = n a_n \lambda^{-1} \text{ and } (n-1)a_{n+1} = n a_n \lambda.$$

Hence $a_n = 0$, which is a contradiction. This case does not occur. \square

3.2 Rank 1 free $U(\mathbb{C}L_0 \oplus \mathbb{C}W_0)$ -modules. In this subsection, we will classify the modules over the $W(2,2)$ algebra \mathcal{W} such that the modules are free $U(\mathbb{C}L_0 \oplus \mathbb{C}W_0)$ -modules of rank 1. Before present the main result, we first construct some modules with this property.

Fix any $\lambda \in \mathbb{C}^*$ and $\alpha \in \mathbb{C}$. For any $k \in \mathbb{N}$ and $n \in \mathbb{Z}$, define the following polynomials

$$h_{n,k;\alpha}(t) = n t^k - n(n-1)\alpha \frac{t^k - \alpha^k}{t - \alpha} \in \mathbb{C}[t] \quad (3.3)$$

and let \mathcal{H}_α be the set of families of polynomials given by

$$\mathcal{H}_\alpha = \left\{ \{h_n(t)\}_{n \in \mathbb{Z}} \mid h_n(t) = \sum_{i=0}^{+\infty} \xi_i h_{n,i;\alpha}(t) \in \mathbb{C}[t], \xi_i \in \mathbb{C} \right\}. \quad (3.4)$$

Note that ξ_i 's are independent of the choice of n . In particular, we have

$$\mathcal{H}_0 = \left\{ \{n h(t)\}_{n \in \mathbb{Z}} \mid h(t) \in \mathbb{C}[t] \right\}. \quad (3.5)$$

For any $\mathbf{h} = \{h_n(t)\} \in \mathcal{H}_\alpha$, denote by $\Omega(\lambda, \alpha, \mathbf{h}) = \mathbb{C}[t, s]$ the polynomial algebra over \mathbb{C} . We define the action of \mathcal{W} on $\Omega(\lambda, \alpha, \mathbf{h})$ as follows

$$C_1(f(t, s)) = C_2(f(t, s)) = 0, \quad W_m(f(t, s)) = \lambda^m (t - m\alpha) f(t, s - m) \quad (3.6)$$

and

$$L_m(f(t, s)) = \lambda^m (s + h_m(t)) f(t, s - m) - m \lambda^m (t - m\alpha) \frac{\partial}{\partial t} (f(t, s - m)). \quad (3.7)$$

Then we have

Proposition 3.1. $\Omega(\lambda, \alpha, \mathbf{h})$ is a \mathcal{W} -module for any $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$ and $\mathbf{h} = \{h_n(t)\} \in \mathcal{H}_\alpha$ under the actions of (3.6) and (3.7) and $\Omega(\lambda, \alpha, \mathbf{h})$ is free of rank 1 as a module over $U(\mathbb{C}L_0 \oplus \mathbb{C}W_0)$. Moreover, we have that $\Omega(\lambda, \alpha, \mathbf{h})$ is simple if and only if $\alpha \neq 0$.

Proof. For any $m, n \in \mathbb{Z}$, we have that

$$\begin{aligned} L_m L_n(f(t, s)) &= L_m \left(\lambda^n (s + h_n(t)) f(t, s - n) - n \lambda^n (t - n\alpha) \frac{\partial}{\partial t} f(t, s - n) \right) \\ &= \lambda^{m+n} (s + h_m(t)) (s - m + h_n(t)) f(t, s - m - n) \\ &\quad - n \lambda^{m+n} (s + h_m(t)) (t - n\alpha) \frac{\partial}{\partial t} f(t, s - m - n) \\ &\quad - m \lambda^{m+n} (t - m\alpha) f(t, s - m - n) \frac{\partial}{\partial t} h_n(t) \\ &\quad - m \lambda^{m+n} (t - m\alpha) (s - m + h_n(t)) \frac{\partial}{\partial t} f(t, s - m - n) \\ &\quad + mn \lambda^{m+n} (t - m\alpha) \frac{\partial}{\partial t} f(t, s - m - n) \\ &\quad + mn \lambda^{m+n} (t - m\alpha) (t - n\alpha) \frac{\partial^2}{\partial t^2} f(t, s - m - n), \end{aligned}$$

$$\begin{aligned} L_m W_n(f(t, s)) &= L_m (\lambda^n (t - n\alpha) f(t, s - n)) \\ &= \lambda^{m+n} (t - n\alpha) (s + h_m(t)) f(t, s - m - n) - m \lambda^{m+n} (t - m\alpha) f(t, s - m - n) \\ &\quad - m \lambda^{m+n} (t - m\alpha) (t - n\alpha) \frac{\partial}{\partial t} (f(t, s - m - n)) \end{aligned}$$

and

$$\begin{aligned} W_n L_m(f(t, s)) &= W_n \left(\lambda^m (s + h_m(t)) f(t, s - m) - m \lambda^m (t - m\alpha) \frac{\partial}{\partial t} (f(t, s - m)) \right) \\ &= \lambda^{m+n} (t - n\alpha) (s - n + h_m(t)) f(t, s - m - n) \\ &\quad - m \lambda^{m+n} (t - n\alpha) (t - m\alpha) \frac{\partial}{\partial t} (f(t, s - m - n)). \end{aligned}$$

Hence for any $m, n \in \mathbb{Z}$, we have

$$\begin{aligned} [L_m, L_n](f(t, s)) &= \lambda^{m+n} \left((s + h_m(t))(s - m + h_n(t)) - (s + h_n(t))(s - n + h_m(t)) \right) f(t, s - m - n) \\ &\quad + \lambda^{m+n} \left(n(t - n\alpha) \frac{\partial}{\partial t} h_m(t) - m(t - m\alpha) \frac{\partial}{\partial t} h_n(t) \right) f(t, s - m - n) \\ &\quad + \lambda^{m+n} \left(m(s + h_n(t))(t - m\alpha) - n(s + h_m(t))(t - n\alpha) \right) \frac{\partial}{\partial t} f(t, s - m - n) \\ &\quad + \lambda^{m+n} \left(n(t - n\alpha)(s - n + h_m(t)) - m(t - m\alpha)(s - m + h_n(t)) \right) \frac{\partial}{\partial t} f(t, s - m - n) \\ &\quad + mn \lambda^{m+n} \left((t - m\alpha) - (t - n\alpha) \right) \frac{\partial}{\partial t} f(t, s - m - n) \\ &= (n - m) \lambda^{m+n} \left(s + (m + n)t^k - (m + n)(m + n - 1)\alpha \frac{t^k - \alpha^k}{t - \alpha} \right) f(t, s - m - n) \\ &\quad - (n - m)(m + n) \lambda^{m+n} (t - (m + n)\alpha) \frac{\partial}{\partial t} f(t, s - m - n) \\ &= (n - m) L_{m+n}(f(t, s)), \end{aligned}$$

where we use the following identity

$$n(t - n\alpha)h'_m(t) - m(t - m\alpha)h'_n(t) = -(n - m)m n \alpha \frac{t^k - \alpha^k}{t - \alpha}, \quad \forall m, n \in \mathbb{Z}.$$

Similarly, we can deduce that

$$\begin{aligned} [L_m, W_n](f(t, s)) &= n\lambda^{m+n}(t - n\alpha)f(t, s - m - n) - m\lambda^{m+n}(t - m\alpha)f(t, s - m - n) \\ &= (n - m)\lambda^{m+n}(t - (m + n)\alpha)f(t, s - m - n) \\ &= (n - m)W_{m+n}(f(t, s)). \end{aligned}$$

Finally, for any $m, n \in \mathbb{Z}$, we have that

$$W_m W_n(f(t, s)) = \lambda^{m+n}(t - n\alpha)(t - m\alpha)f(t, s - m - n), \quad (3.8)$$

hence $[W_m, W_n](f(t, s)) = 0$.

For the simplicity, we have

Case 1. $\alpha \neq 0$.

Suppose that N is a nonzero submodule of $\Omega(\lambda, \alpha, \mathbf{h})$. Let $f(t, s)$ be a nonzero polynomial in N with the smallest $\deg_s(f(t, s))$. By (3.8), we have

$$(2W_0 W_m - W_{m+1} W_{-1} - W_{m-1} W_1)(f(t, s)) = 2\lambda^m \alpha^2 f(t, s - m),$$

forcing $f(t, s - m) \in N$. This implies that $\deg_s(f(t, s)) = 0$, i.e., $f(t, s) \in \mathbb{C}[t]$. Denote $f(t) = f(t, s)$. From the actions of W_0 and L_0 , we see that $f(t)\mathbb{C}[t, s] \subseteq N$. Hence by (3.7) we get $(t - m\alpha)f'(t) \in N$ for all $m \in \mathbb{Z}^*$. This immediately gives that $N = \mathbb{C}[t, s]$, i.e., $\Omega(\lambda, \alpha, \mathbf{h})$ is simple.

Case 2. $\alpha = 0$.

It is clear that for any $i \in \mathbb{Z}_+$, $t^i \mathbb{C}[t, s]$ is a submodule of $\Omega(\lambda, 0, \mathbf{h})$ for all $\lambda \in \mathbb{C}^*$ and $\mathbf{h} = \{nh(t)\}_{n \in \mathbb{Z}} \in \mathcal{H}_0$ with $h(t) \in \mathbb{C}[t]$. Moreover, the quotient module $t^i \mathbb{C}[t, s]/t^{i+1} \mathbb{C}[t, s]$ is simple if and only if $h(0) \neq i$. Indeed, for any $0 \neq t^i g(s) \in t^i \mathbb{C}[t, s]/t^{i+1} \mathbb{C}[t, s]$, we have $W_m(t^i g(s)) = 0$ and

$$\begin{aligned} L_m(t^i g(s)) &\equiv \lambda^m(s + mh(0))t^i g(s - m) - \lambda^m m i t^i g(s - m) \\ &\equiv \lambda^m(s + m(h(0) - i))t^i g(s - m), \quad \text{mod } t^{i+1} \mathbb{C}[t, s]. \end{aligned}$$

Hence $t^i \mathbb{C}[t, s]/t^{i+1} \mathbb{C}[t, s] \cong \Omega_{\mathcal{W}}(\lambda, i - h(0))$ as \mathcal{W} -modules. \square

The following is our main result of this subsection:

Theorem 4. Let M be a $U(\mathcal{W})$ -module such that the restriction to $U(\mathbb{C}L_0 \oplus \mathbb{C}W_0)$ is free of rank 1. Then $M \cong \Omega(\lambda, \alpha, \mathbf{h})$ for some $\alpha \in \mathbb{C}$, $\lambda \in \mathbb{C}^*$ and $\mathbf{h} = \{h_n(t)\} \in \mathcal{H}_\alpha$.

Proof. Denote the action of C_1, C_2 by c_1, c_2 , respectively. Assume that $M = U(\mathbb{C}L_0 \oplus \mathbb{C}W_0)v$ for $v \in M$. We divide the proof into several claims.

Claim 1. For any $u \in M$, $i \in \mathbb{Z}_+$ and $m \in \mathbb{Z}$, we have

$$W_m(W_0^i u) = W_0^i W_m u \text{ and } W_m(L_0^i u) = (L_0 - m)^i W_m u. \quad (3.9)$$

Claim 2. $W_m(v) \neq 0$ for all $m \in \mathbb{Z}$.

Suppose on the contrary that $W_m(v) = 0$ for some $m \in \mathbb{Z}^*$. By (3.9) we know that $W_m(M) = 0$. Now for any $n \neq 0, 2m$, we have

$$W_n M = (2m - n)^{-1} [L_{n-m}, W_m] M = 0.$$

Moreover, we have

$$W_{2m} M = (4m)^{-1} [L_{-m}, W_{3m}] M = 0.$$

Hence, for any $n \in \mathbb{N}$, we get

$$-2nW_0v + \frac{n^3 - n}{12}c_2v = [L_n, W_{-n}]v = 0.$$

This implies that $W_0v = 0$, a contradiction. This claim is true.

Claim 3. $W_m(v) = a_m(W_0)$ for some $a_m \in \mathbb{C}[W_0]$.

Assume that

$$W_m(v) = \sum_{i=0}^{k_m} b_{m,i} L_0^i v$$

where $b_{m,i} = b_{m,i}(W_0) \in \mathbb{C}[W_0]$ and $b_{m,k_m} \neq 0$. For all $mn \leq 0$, we have

$$\begin{aligned} 0 &= [W_m, W_n](v) = W_m W_n(v) - W_n W_m(v) \\ &= \left(\sum_{i=0}^{k_n} b_{n,i} (L_0 - m)^i \right) \left(\sum_{i=0}^{k_m} b_{m,i} L_0^i \right) v - \left(\sum_{i=0}^{k_n} b_{n,i} L_0^i \right) \left(\sum_{i=0}^{k_m} b_{m,i} (L_0 - n)^i \right) v \\ &= b_{m,k_m} b_{n,k_n} (nk_m - mk_n) L_0^{k_m+k_n-1} v + \text{lower-degree terms w.r.t. } L_0^i v. \end{aligned}$$

Hence $k_m = 0$ for all $m \in \mathbb{Z}$, i.e., $W_m(v) = a_m(W_0)v$ with $a_m(W_0) \in \mathbb{C}[W_0]$. In particular, $a_0(W_0) = W_0$.

Claim 4. For any $u \in M$, $i \in \mathbb{Z}_+$ and $m \in \mathbb{Z}$, we have

$$L_m(L_0^i u) = (L_0 - m)^i L_m u \text{ and } L_m(W_0^i u) = W_0^i L_m u - imW_0^{i-1} W_m u. \quad (3.10)$$

We just prove the second relation by induction on i . If $i = 0$, there is nothing to prove. Now assume the relation holds for i , then we have

$$\begin{aligned} L_m(W_0^{i+1} u) &= [L_m, W_0] W_0^i u + W_0 (L_m(W_0^i u)) \\ &= -mW_m W_0^i u + W_0 (W_0^i L_m u - imW_0^{i-1} W_m u) \\ &= W_0^{i+1} L_m u - (i+1)mW_0^i W_m u. \end{aligned}$$

Moreover, the second relation is equivalent to

$$L_m(f(W_0)u) = f(W_0)L_m u - mf'(W_0)W_m u \quad \forall f(W_0) \in \mathbb{C}[W_0]. \quad (3.11)$$

Claim 5. $\deg(a_m(W_0)) = 1$ for all $m \in \mathbb{Z}$.

Now assume $L_m v = g_m(L_0, W_0)v$ for some polynomial $g_m(L_0, W_0) \in \mathbb{C}[L_0, W_0]$. Then the relation

$$[L_m, W_n](v) = (n - m)W_{m+n}(v) + \delta_{m+n,0} \frac{m^3 - m}{12} c_2 v \quad (3.12)$$

implies

$$\begin{aligned} \mathbb{C}[W_0]v &\ni (n - m)W_{m+n}(v) + \delta_{m+n,0} \frac{m^3 - m}{12} c_2 v \\ &= L_m(a_n(W_0)v) - W_n(g_m(L_0, W_0)v) \\ &= a_n(W_0)L_m v - m a'_n(W_0)W_m v - g_m(L_0 - n, W_0)a_n(W_0)v \\ &= a_n(W_0)(g_m(L_0, W_0) - g_m(L_0 - n, W_0))v - m a'_n(W_0)a_m(W_0)v. \end{aligned}$$

Now we get $\deg_{L_0}(g_m(L_0, W_0) - g_m(L_0 - n, W_0)) = 0$. It follows that $\deg_{L_0} g_m(L_0, W_0) \leq 1$ for all $m \in \mathbb{Z}$, i.e., $g_m(L_0, W_0) = b_m(W_0)L_0 + d_m(W_0)$ for some $b_m(W_0), d_m(W_0) \in \mathbb{C}[W_0]$. Now (3.12) is equivalent to

$$n a_n(W_0)b_m(W_0) - m a'_n(W_0)a_m(W_0) = (n - m)a_{m+n}(W_0) + \delta_{m+n,0} \frac{m^3 - m}{12} c_2. \quad (3.13)$$

Taking $m = n$ in (3.13), we have $b_m(W_0) = a'_m(W_0)$. Now let $m = -n$ in (3.13), then we can obtain $a_n(W_0)a_{-n}(W_0) = W_0^2 - \frac{n^2-1}{12}c_2 W_0 + x_n$ for $n \in \mathbb{Z}^*$ and some $x_n \in \mathbb{C}$. This implies that $\deg(a_m(W_0)) \leq 2$ for all $m \in \mathbb{Z}$.

If there are $m \neq n \in \mathbb{Z}$ such that $\deg(a_m(W_0)) = \deg(a_n(W_0)) = 0$ (it is clear that $m + n \neq 0$), then

$$a_{m+n}(W_0) = (n - m)^{-1} (n a_n(W_0)a'_m(W_0) - m a'_n(W_0)a_m(W_0)) = 0, \quad (3.14)$$

which is a contradiction.

If $\deg(a_m(W_0)) = 0$ for some $m \in \mathbb{Z}^*$, then $\deg(a_{2m}(W_0)) = 1$. By taking $n = 2m$ in (3.13) we get

$$a_{3m}(W_0) = 2a_{2m}(W_0)a'_m(W_0) - a'_{2m}(W_0)a_m(W_0) = -a'_{2m}(W_0)a_m(W_0), \quad (3.15)$$

hence $\deg(a_{3m}(W_0)) = 0$, which is also a contradiction. Now we have $\deg(a_m(W_0)) = 1$ for all $m \in \mathbb{Z}$. This claim follows.

Claim 6. $a_m(W_0) = \lambda^m(W_0 - m\alpha)$ for some $\lambda \in \mathbb{C}^*$ and $\alpha \in \mathbb{C}$; and $c_2 = 0$.

From Claim 5 we may assume $a_m(W_0) = a_{m,1}W_0 + a_{m,0}$ for $a_{m,1}, a_{m,0} \in \mathbb{C}$ with $a_{m,1}a_{-m,1} = 1$, $a_{0,1} = 1$ and $a_{0,0} = 0$. Hence $b_m(W_0) = a'_m(W_0) = a_{m,1}$.

The relation (3.13) implies that

$$a_{n,1}a_{m,1} = a_{m+n,1}, \quad \forall m \neq n \quad (3.16)$$

and

$$n a_{n,0}a_{m,1} - m a_{n,1}a_{m,0} = (n - m)a_{m+n,0} + \delta_{m+n,0} \frac{m^3 - m}{12} c_2. \quad (3.17)$$

Then (3.16) gives that $a_{m,1} = a_{1,1}^m$ for all $m \in \mathbb{Z}$. Now (3.17) gives that $a_{m,0} = m a_{1,0} a_{1,1}^{m-1}$ and $c_2 = 0$. Set $\lambda = a_{1,1}$ and $\alpha = -a_{1,0}/a_{1,1}$, and this claim follows.

Claim 7. $g_n(L_0, W_0) = \lambda^n(L_0 + h_n(W_0))$, where $\{h_n(t)\} \in \mathcal{H}_\alpha$ (see (3.4)); and $c_1 = 0$.

The equation

$$[L_m, L_n](v) = (n - m)L_{m+n}(v) + \delta_{m+n,0} \frac{m^3 - m}{12} c_1 v \quad (3.18)$$

is equivalent to

$$nb_m d_n + nd'_m a_n - md_m b_n - ma_m d'_n = (n - m)d_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} c_1. \quad (3.19)$$

In particular, by taking $m = -n$ and noticing that $d_0(W_0) = 0$, we have that

$$(d_{-n} a_n + d_n a_{-n})' = \frac{1 - n^2}{12} c_1. \quad (3.20)$$

For convenience, we denote $F_m = \lambda^{-m} d_m(W_0) \in \mathbb{C}[W_0]$, then (3.19) is equivalent to

$$nF_n + nF'_m(W_0 - n\alpha) - mF_m - mF'_n(W_0 - m\alpha) = (n - m)F_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} c_1. \quad (3.21)$$

It is clear that

$$\deg_{W_0} F_n \leq \max\{\deg_{W_0} F_{-2}, \deg_{W_0} F_{-1}, \deg_{W_0} F_0, \deg_{W_0} F_1, \deg_{W_0} F_2\}$$

for $n \in \mathbb{Z}$, i.e., the degrees of all $F_n, n \in \mathbb{Z}$ are bounded. Choose $k \in \mathbb{Z}_+$ such that $\deg F_n \leq k$ for all $n \in \mathbb{Z}$. Let f_n be the coefficient of W_0^k of $F_n(W_0)$.

Checking the coefficients of W_0^k in (3.21), we have

$$(n - km)f_n + (kn - m)f_m = (n - m)f_{m+n} + \delta_{k,0} \delta_{m+n,0} \frac{m^3 - m}{12} c_1. \quad (3.22)$$

Case 1. $k = 0$.

In this case, we have

$$nf_n - mf_m = (n - m)f_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} c_1.$$

If $m + n \neq 0$, we have $(n - m)f_{m+n} = nf_n - mf_m$, hence $f_n = (n - 1)f_2 - (n - 2)f_1$ for $n \in \mathbb{Z}^*$. For $m = -n \neq 0$, noticing that $f_0 = 0$, we have

$$\frac{1 - n^2}{12} c_1 = (n - 1)f_2 - (n - 2)f_1 + (-n - 1)f_2 - (-n - 2)f_1.$$

We get $c_1 = 0$ and $f_2 = 2f_1$. Now we have $f_n = nf_1$.

Case 2. $k \geq 1$.

Taking $n = 2$ and $m = 1$ in (3.22), we have

$$f_3 = (2 - k)f_2 + (2k - 1)f_1.$$

Taking $n = 3$ and $m = 1$ in (3.22), we have

$$\begin{aligned} 2f_4 &= (3 - k)f_3 + (3k - 1)f_1 \\ &= (3 - k)(2 - k)f_2 + (10k - 2k^2 - 4)f_1 \end{aligned}$$

Taking $n = 4$ and $m = 1$ in (3.22), we have

$$\begin{aligned} 6f_5 &= (4-k)2f_4 + 2(4k-1)f_1 \\ &= (-k^3 + 9k^2 - 26k + 24)f_2 + 2(k^3 - 9k^2 + 26k - 9)f_1 \end{aligned}$$

Taking $n = 3$ and $m = 2$ in (3.22), we have

$$\begin{aligned} 6f_5 &= 6(3-2k)f_3 + 6(3k-2)f_2 \\ &= 12(k^2 - 2k + 2)f_2 + 6(8k - 4k^2 - 3)f_1 \end{aligned}$$

Calculating the difference of the last two equations we get $f_2 = 2f_1$ and by induction $f_n = nf_1$ for all $n \in \mathbb{Z}$.

From Proposition 3.1, we see that the family of polynomials in W_0

$$h_{n,k} = nW_0^k - n(n-1)\alpha \frac{W_0^k - \alpha^k}{W_0 - \alpha}$$

satisfy the equation (3.21) with c_1 replaced with 0. Then there exists $\xi_k \in \mathbb{C}$ such that $\deg(f_n - \xi_k h_{n,k}) \leq k-1$ for all $n \in \mathbb{Z}$. Noticing that the polynomials $f_n - \xi_k h_{n,k}$ again satisfy the equation (3.21). Replacing f_n with $f_n - \xi_k h_{n,k}$ in the arguments of Case 2 and repeating this process, we may find $\xi_1, \dots, \xi_k \in \mathbb{C}$ such that the polynomials $f_n - \sum_{i=1}^k \xi_i h_{n,i}$ are all constant and satisfy the equation (3.21). Replace f_n with $f_n - \sum_{i=1}^k \xi_i h_{n,i}$ in the arguments of Case 1, we have $c_1 = 0$ and

$$f_n - \sum_{i=1}^k \xi_i h_{n,i} = n(f_1 - \sum_{i=1}^k \xi_i h_{1,i}) = \xi_0 h_{n,0}$$

for some $\xi_0 \in \mathbb{C}$. As a result, we obtain that $f_n = \sum_{i=0}^k \xi_i h_{n,i}$, as desired.

Finally, Claim 1, 4, 6, 7 together indicate that $M \cong \Omega(\lambda, \alpha, \mathbf{h})$, where $\mathbf{h} = \{h_n(t)\} \in \mathcal{H}_\alpha$ is as given in Claim 7. Since simplicity follows from Proposition 3.1 directly, we have completed the proof. \square

Remark 3.2. We remark that the simple subquotients of the \mathcal{W} -module $\Omega_{\mathcal{W}}(\lambda, \alpha)$ and $\Omega(\lambda, \alpha, \mathbf{h})$ are all non-weight modules and neither of the actions of L_m is locally nilpotent in these modules. Hence they are new classes of modules over \mathcal{W} .

References

- [ACKP] E. Arbarello, C. De Concini, V. G. Kac and C. Procesi, Moduli spaces of curves and representation theory, *Comm. Math. Phys.* 117(1) (1988) 1–36.
- [B] Y. Billig, Representations of the twisted Heisenberg-Virasoro algebra at level zero. *Canad. Math. Bull.* 46(4) (2003) 529–537.
- [BM] P. Batra and V. Mazorchuk, Blocks and modules for Whittaker pairs. *J. Pure Appl. Algebra*, 215(2011) 1552–1568.
- [CG] H. Chen and X. Guo New simple modules for the HeisenbergCVirasoroalgebra. *J. Algebra*, 390(2013), 77–86.

- [CGZ] H. Chen, X. Guo and K. Zhao, Tensor product weight modules over the Virasoro algebra. *J. Lond. Math. Soc.*, 83(3)(2013) : 829–844.
- [CM] C. Conley and C. Martin, A family of irreducible representations of Witt Lie algebra with infinite-dimensional weight spaces. *Compos. Math.*, 128(2)(2001) 152–175.
- [GLZ] X. Guo, R. Lu and K. Zhao, Simple Harish-Chandra modules, intermediate series modules, and Verma modules over the loop-Virasoro algebra. *Forum Mathematicum*, 23 (2011), 1029–1052.
- [K] V.G. Kac, Some problems of infinite-dimensional Lie algebras and their representations. *Lecture Notes in Mathematics*, 993, (1982)117–126. Berlin, Heidelberg, New York: Springer.
- [KR] V. Kac and A. Raina, Bombay lectures on highest weight representations of infinite dimensional Lie algebras. World Sci., Singapore, 1987.
- [LGZ] R. Lü, X. Guo and K. Zhao, Irreducible modules over the Virasoro algebra, *Doc. Math.*, 16 (2011), 709–721.
- [LLZ] G. Liu, R. Lü and K. Zhao, A class of simple weight Virasoro modules. Preprint, arXiv:1211.0998.
- [LiZ] D. Liu and L. Zhu, Classification of irreducible weight modules over W -algebra $W(2,2)$, *J. Math. Phys.* 49 (2008), no. 11, 113503, 6 pp.
- [LZ1] R. Lü and K. Zhao, Irreducible Virasoro modules from irreducible Weyl modules. Preprint, arXiv:1209.3746.
- [LZ2] R. Lü and K. Zhao, A family of simple weight modules over the Virasoro algebra. Preprint, arXiv:1303.0702.
- [LZ3] R. Lü and K. Zhao; Classification of irreducible weight modules over the twisted Heisenberg-Virasoro algebra. *Commun. Contemp. Math.*, **12** (2010), no. 2, 183–205.
- [M] Classification of Harish-Chandra modules over the Virasoro Lie algebra. *Invent. Math.*, 107(2)(1992) 225–234.
- [MW] V. Mazorchuk and E. Weisner, Simple Virasoro modules induced from codimension one subalgebras of the positive part. *Proc. Amer. Math. Soc.*, in press, arXiv:1209.1691,
- [MZ] V. Mazorchuk and K. Zhao, Simple Virasoro modules which are locally finite over a positive part. *Selecta Math. New Ser.*, DOI 10.1007/s00029-013-0140-8.
- [N] J. Nilsson, Simple \mathfrak{sl}_{n+1} -module structures on $U(\mathfrak{h})$. Preprint, arXiv:1312.5499.
- [OW] M. Ondrus and E. Wiesner, Whittaker Modules for the Virasoro Algebra, *J. Algebra Appl.*, 8(2009) 363–377.
- [TZ1] H. Tan and K. Zhao, Irreducible modules from tensor products. Preprint, arXiv:1301.2131.

- [TZ2] H. Tan and K. Zhao, Irreducible modules from tensor products (II). *J. Algebra*, 394(2013) 357-373.
- [TZ3] H. Tan and K. Zhao, Irreducible modules over Witt algebras \mathcal{W}_n and over $\mathfrak{sl}_{n+1}(\mathbb{C})$. Preprint, arXiv:1312.5539.
- [TZ4] H. Tan and K. Zhao, \mathcal{W}_n -module structures on $U(\mathfrak{h})$. Preprint, arXiv:1401.1120.
- [Z] H. Zhang, A class of representations over the Virasoro algebra, *J. Algebra*, 190(1997) 1–10.
- [ZD] W. Zhang and C. Dong, W-algebra $W(2, 2)$ and the vertex operator algebra $L(1 = 2; 0) - L(1 = 2; 0)$, *Comm. Math. Phys.* 285(2009), 991–1004.